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Whitney constants and approximation of m-quasi-linear forms by m-linear forms

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Abstract

We discuss some relations between Whitney constants $w_m(B_X, Y)$ for bounded functions from, the unit ball of a real normed space X into another real normed space Y. In particular, we generalize a result of Tsar'kov that

$$w_m^l(B_X, Y) \sim n^{(m-1)/2} \quad \text{for } X = l_2^n \text{and for any} Y$$

to any n-dimensional X (here w_m^l denotes linearized Whitney constant).

MSC: 41A10

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1. Introduction

X and Y denote real normed spaces throughout. The closed ball with center z and radius r is denoted by $B(z, r)$, and we write $B(0, r) = B(r)$ and $B(1) = B$ (or B_E, when we need to specify the space); S_X denotes the unit sphere of X. The Banach–Mazur distance between X and Y is denoted by $d(X, Y)$. If X has non-trivial type q, we denote its q-type constant by $T_q(X)$. For a set A, $\mathcal{B}(A, Y)$ denotes the normed space of bounded functions from A into

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Y with the supremum norm. For a natural m, we denote by $L^m(X,Y)$ the normed space of bounded m-linear forms Ψ from X^m into Y with the usual norm

$$\|\Psi\| = \sup\{\|\Psi(x_1,\ldots,x_m)\| : \{x_i\} \subset B_X\}$$

and by $\mathcal{P}_m(X,Y)$ the linear space of polynomials of total degree at most m, that is, $p(x) = \sum_{i=0}^{m} \Psi_i(x,\ldots,x)$ where $\Psi_i \in L^i(X,Y)$. For $A \subset X$ and $f \in \mathcal{B}(A,Y)$, we set

$$E_m(f) = E_m(f; A, Y) = \inf_{p\in\mathcal{P}_{m-1}} \sup_{x\in A} \|f(x) - p(x)\|$$

and

$$\omega_m(f) = \omega_m(f; A, Y) = \sup\left\{\|\Delta_h^m(f;x)\| : [x, x+mh] \subset A\right\},$$

where

$$\Delta_h^m(f;x) = \sum_{i=0}^{m} (-1)^{m-i}\binom{m}{i} f(x+ih).$$

We then define a *Whitney constant* $w_m(A,Y)$ by

$$w_m(A,Y) = \sup\{E_m(f) : f \in \mathcal{B}(A,Y) \text{ and } \omega_m(f) \leqslant 1\}$$

and a *linearized Whitney constant* $w_m^l(A,Y)$ by

$$w_m^l(A,Y) = \inf_L \sup\left\{\|f - Lf\|_{\mathcal{B}(A,Y)} : f \in \mathcal{B}(A,Y) \text{ and } \omega_m(f) \leqslant 1\right\},$$

where L runs through all linear operators $L : \mathcal{B}(A,Y) \longrightarrow \mathcal{P}_{m-1}(X,Y)$. (The last quantity is important because of the (computational) universality of linear approximation methods.) We shall write $w_m(X,Y)$ in place of $w_m(B_X,Y)$.

Tsar'kov [T] proved that $w_m^l(l_2^n, Y) \sim n^{(m-1)/2}$ for any Y. For $m = 2$, the author [V] (among other results) obtained, in fact, the following.

Proposition 1 (*Vestfrid [V, Proposition 3.2]*). *Let* $\dim X = n$ *and* $B_X(r) \subseteq A \subset B_X(R)$ *be star-shaped with respect to the origin. Then there is an absolute constant k such that for every* $1 < q \leqslant 2$ *and any* Y,

$$w_2^l(A,Y) \leqslant \frac{k}{q-1}(1 + |\log(q-1)| + \log T_q(X))d(l_1^n, X)R/r.$$

As it may be anticipated, approximation methods for individual functions can be better than a linear one. Brudnyi and Kalton [BK] showed, for example, that $w_m(X, \mathbf{R}) \leqslant Cn^{(m-2)/2}\log(n+1)$ for $m \geqslant 2$ and for any n-dimensional X and that $w_m(l_p^n, \mathbf{R}) \leqslant Cn^{(m-3)/2}\log(n+1)$ for $m \geqslant 3$ and $2 \leqslant p < \infty$. For technical reason, they introduced scalar m-quasi-linear functions and heavily used its approximation by m-linear forms. It seems us helpful to extend this concept to the multi-dimensional case as follows.

Definition 2 (*cf. Brudnyi and Kalton [BK, p. 193]*). Let A be a subset of X with $0 \in A$. Let $K \geqslant 0$. A map $f : A^m \longrightarrow Y$ is said to be (m,K)-quasi-linear if it satisfies the following two conditions:

(1) $f(x_1, \ldots, x_m) = 0$, whenever at least one $x_i = 0$;

(2) for any $1 \leqslant j \leqslant m$ and any $\{x_i\}_{i \neq j} \subset A$, the map $f_j(x_1, \ldots x_{j-1}, x_{j+1}, \ldots, x_m)(x) = f(x_1, \ldots x_{j-1}, x, x_{j+1}, \ldots, x_m)$ satisfies $\omega_2\big(f_j(x_1, \ldots x_{j-1}, x_{j+1}, \ldots, x_m);\ A, Y\big) \leqslant K$.

f is said to be homogeneously (m, K)-quasi-linear if its domain is the whole X^m and it is (m, K)-quasi-linear on B_X^m and homogeneous in each variable separately.

f is called an m-quasi-linear map if it is (m, K)-quasi-linear for some $K \geqslant 0$.

We denote by $QL^m(A, Y)$ the linear space of bounded m-quasi-linear maps from A into Y.

We shall abbreviate "quasi-linear" by QL.

In this paper, we obtain some relations between Whitney constants, which yields, in particular, a generalization of Tsar'kov's result to any n-dimensional X (Theorem 6 and Remark 7) and, in a combination with results of Brudnyi and Kalton, gives a sharp estimate $w_2\left(l_p^n, \left(l_p^n\right)^*\right) \sim w_3(l_p^n, \mathbf{R}) \sim \log(n+1)$, if $p = 1$ or $2 \leqslant p < \infty$ (see Remark 10(ii)). One of the keys is Proposition 3 on approximation of bounded m-quasi-linear forms by m-linear forms.

2. Results

Proposition 3. *Let $B_X(r) \subseteq A \subset B_X(R)$ be star-shaped with respect to the origin. Let $f\colon A^m \longrightarrow Y$ be a bounded (m, K)-QL map. Then there are a constant C_m, depending only on m, and a continuous m-linear form $\Psi_m\colon X^m \longrightarrow Y$ such that*

$$\|f(x_1, \ldots, x_m) - \Psi_m(x_1, \ldots, x_m)\| \leqslant C_m w_2(A, Y) \prod_{i=1}^{m-1} w_2\left(A,\ L^i(X, Y)\right) K (R/r)^{m-1} \qquad (1)$$

for every $\{x_i\} \subset A$.

Proof. We shall prove by induction on m. By the definitions, the proposition holds for $m = 1$ (with any $C_1 > 1$). Assume it holds for some $m \geqslant 1$, and let f be an $(m+1, K)$-QL map. Then for every $\{x_i\}_{i \leqslant m} \subset A$, $\omega_2\big(f_{m+1}(x_1, \ldots, x_m); A, Y\big) \leqslant K$. By the definition of $w_2(A, Y)$, there is a linear bounded operator $F(x_1, \ldots, x_m)\colon X \longrightarrow Y$ for every m-tuple $(x_1, \ldots, x_m) \in A^m$ such that $F(x_1, \ldots, x_m) \equiv 0$, whenever at least one $x_i = 0$ and

$$\|f_{m+1}(x_1, \ldots, x_m)(x) - F(x_1, \ldots, x_m)x\| \leqslant w_2(A, Y) K \qquad (2)$$

for all $x \in A$.

Now regard $F(x_1, \ldots, x_m)$ as a map from A^m into $L(X, Y)$. Let $1 \leqslant j \leqslant m$ and $\{x_i\}_{i \neq j} \subset A$, and denote $F_j = F_j(x_1, \ldots x_{j-1}, x_{j+1}, \ldots, x_m)$. Then for every $u, x \in A$ and $h \in X$

with $[u, u+2h] \subset A$, we have by (2)

$$\begin{aligned}\left\|\left(\Delta_h^2(F_j; u)\right)(x)\right\| \leqslant\ & \|f(x_1, \dots x_{j-1}, u, x_{j+1}, \dots, x_m, x) \\ & - F(x_1, \dots x_{j-1}, u, x_{j+1}, \dots, x_m)x\| \\ & + 2\|f(x_1, \dots x_{j-1}, u+h, x_{j+1}, \dots, x_m, x) \\ & - F(x_1, \dots x_{j-1}, u+h, x_{j+1}, \dots, x_m)x\| \\ & + \|f(x_1, \dots x_{j-1}, u+2h, x_{j+1}, \dots, x_m, x) \\ & - F(x_1, \dots x_{j-1}, u+2h, x_{j+1}, \dots, x_m)x\| \\ & + \left\|\Delta_h^2\big(f_j(x_1, \dots x_{j-1}, x_{j+1}, \dots, x_m, x);\ u\big)\right\| \\ \leqslant\ & 4w_2(A, Y)K + K.\end{aligned}$$

Thus, F is $\big(m,\ 5w_2(A, Y)K/r\big)$-QL, and by the induction hypothesis, there is a continuous m-linear form $\Psi_m\colon X^m \longrightarrow L(X, Y)$ such that

$$\begin{aligned}&\|F(x_1, \dots, x_m) - \Psi_m(x_1, \dots, x_m)\| \\ &\leqslant 5C_m w_2(A, Y)K/r \prod_{i=1}^{m} w_2\left(A,\ L^i(X, Y)\right)(R/r)^{m-1}.\end{aligned} \tag{3}$$

Now set $\Psi_{m+1}(x_1, \dots, x_{m+1}) = \Psi_m(x_1, \dots, x_m)(x_{m+1})$. Then by (2) and (3),

$$\begin{aligned}&\|f(x_1, \dots, x_{m+1}) - \Psi_{m+1}(x_1, \dots, x_{m+1})\| \\ &\leqslant \|f(x_1, \dots, x_{m+1}) - F(x_1, \dots, x_m)x_{m+1}\| \\ &\quad + \|\big(F(x_1, \dots, x_m) - \Psi_m(x_1, \dots, x_m)\big)(x_{m+1})\| \\ &\leqslant w_2(A, Y)K + 5C_m w_2(A, Y) \prod_{i=1}^{m} w\left(A,\ L^i(X, Y)\right) K(R/r)^{m-1}\|x_{m+1}\|/r,\end{aligned}$$

which completes the proof. □

Remark 4. (i) An inspection of the above proof also gives the following:

Let $B_X(r) \subseteq A \subset B_X(R)$ be star-shaped with respect to the origin. Then there are a constant C_m, depending only on m, and a linear projector $L\colon QL^m(A, Y) \longrightarrow L^m(X, Y)$ such that for every bounded (m, K)-QL map $f\colon A^m \longrightarrow Y$, we have

$$\begin{aligned}&\|f(x_1, \dots, x_m) - Lf(x_1, \dots, x_m)\| \\ &\leqslant C_m w_2^l(A, Y) \prod_{i=1}^{m-1} w_2^l\left(A,\ L^i(X, Y)\right) K(R/r)^{m-1}\end{aligned}$$

for every $\{x_i\} \subset A$.

(ii) In particular, inequality (1) in Proposition 3 holds for every $\{x_i\} \subset rS_X$. Hence if f is homogeneously (m, K)-QL, we can take $A = B_X$ and then rewrite (1) as

$$\begin{aligned}&\|f(x_1, \dots, x_m) - \Psi_m(x_1, \dots, x_m)\| \\ &\leqslant C_m w_2(X, Y) \prod_{i=1}^{m-1} w_2\left(X,\ L^i(X, Y)\right) \prod_{i=1}^{m} \|x_i\| K\end{aligned}$$

for all $\{x_i\} \subset X$.

Theorem 5. *For any $m \geqslant 2$ there is a constant C_m, depending only on m, such that*

$$w_m(X, Y) \leqslant C_m w_2(X, Y) \prod_{i=1}^{m-2} w_2\left(X,\ L^i(X, Y)\right).$$

Proof. For all integers $0 \leqslant i \leqslant m-1$ and $1 \leqslant j \leqslant m$, choose real numbers c_{ij} satisfying

$$\sum_{j=1}^{m} c_{ij}(j/m)^k = \delta_{ik} \tag{4}$$

for $0 \leqslant i, k \leqslant m-1$.

Let $f \in \mathcal{B}(B_X, Y)$ and $\omega_m(f; B_X, Y) \leqslant 1$. Then for each $x \in S_X$ and $1 \leqslant i \leqslant m-1$ we define $f_i(x) = \sum_{j=1}^{m} c_{ij} f(jx/m)$ and extend f_i to all X to be i-homogeneous (that is, $f_i(tx) = t^i f(x)$ for $t \in \mathbf{R}, x \in X$). We also define $f_0(x) \equiv f(0)$. Then, there is $C = C(m)$ so that

$$\left\| f(x) - \sum_{i=0}^{m-1} f_i(x) \right\| \leqslant C$$

(see, for example, [BK, pp. 169–170]).

Suppose now that $g: X \longrightarrow Y$ is locally bounded and k-homogeneous. Define the separately homogeneous map $G: X^k \longrightarrow Y$ by

$$G(x_1, \ldots, x_k) = \frac{1}{2^k k!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \ldots \varepsilon_k g\left(\sum_{i=1}^{k} \varepsilon_i x_i\right)$$

for $\{x_i\} \subset S_X$ and extend it by homogeneity. Then there is $C_k = C(k)$ so that G is homogeneously $(k,\ C_k\omega_{k+1}(g; B_X, Y))$-quasi-linear (see [BK, Lemma 5.4]). Note also that $G(x, \ldots, x) = g(x)$.

Combining all this with Remark 4(ii) implies the theorem. □

The same proof combined with Remark 4(i) gives us the following generalization of Tsar'kov's result.

Theorem 6. *For any $m \geqslant 2$ there is a constant C_m, depending only on m, such that*

$$w_m^l(X, Y) \leqslant C_m w_2^l(X, Y) \prod_{i=1}^{m-2} w_2^l(X,\ L^i(X, Y)).$$

Remark 7. Theorem 6 combined with Proposition 1 implies, in particular, that for $1 < p < \infty$ there is a constant $C(m, p)$, depending only on m and p, such that

$$w_m^l(l_p^n, Y) \leqslant C(m, p) d(l_1^n, l_p^n)^{m-1}. \tag{5}$$

Thus indeed, Theorem 6 generalizes Tsar'kov's result.

The following counterpart of Theorem 5 is essentially contained in Kalton [K], but the author cannot conclude the result from there; we give the proof for the completeness and convenience of the reader.

Proposition 8. *There is an absolute constant C with the following property*:
Let X be a normed space, and put $X' = (X \oplus X)_p$ for some $p \in [1, \infty]$. Then

$$w_2\,(X,\ L(X, Y)) \leqslant C w_3(X', Y).$$

To prove this we need the next assertion. Kalton [K, the proof of Theorem 2.2] implicitly obtained it for $X = l_2^n,\ Y = \mathbf{R}$, but his argument, actually, yields the more general Lemma 9.

Lemma 9. *There is an absolute constant C with the following property*:
Let $g\colon X \longrightarrow L(X, Y)$ be a 1-homogeneous locally bounded map with

$$\|g(x_1 + x_2) - g(x_1) - g(x_2)\| \leqslant 1, \qquad x_1, x_2 \in B_X.$$

Put $X' = (X \oplus X)_p$ for some $p \in [1, \infty]$. Then for every $\delta > 0$ there is a bounded linear operator $F_\delta\colon X \longrightarrow L(X, Y)$ with

$$\|g(x) - F_\delta x\| \leqslant C \left(w_3(X', Y) + \delta\right) \|x\|$$

for all $x \in X$.

Proof. Put $q(x) = g(x)x\colon X \longrightarrow Y$, and observe that q is 2-homogeneous. Since

$$\Delta_h^3(q; x) = \left(3\Delta_h^2(g; x+h) - \Delta_h^2(g; x)\right)x + 3\Delta_h^2(g; x+h)h$$

for every $x, h \in X$ with $x, x + 3h \in B_X$ we have by homogeneity of g

$$\left\|\Delta_h^3(q; x)\right\| \leqslant 6.$$

Now define $g'(\mathbf{x}) = \frac{1}{6}(0,\ g(x_1))\colon X' \longrightarrow L(X', Y)$ if $\mathbf{x} = (x_1, x_2),\ x_1, x_2 \in X$, and then put $q'(\mathbf{x}) = g'(\mathbf{x})\mathbf{x} = \frac{1}{6} g(x_1)x_2$. Then $\omega_3(q'; B_{X'}, Y) \leqslant 1$ and, by the definition of $w_3(X', Y)$, for every $\delta > 0$ there is a polynomial $p \in P_2(X', Y)$ with $\|q'(\mathbf{x}) - p(\mathbf{x})\| \leqslant w_3(X', Y) + \delta$ on $B_{X'}$.

By the 2-homogeneity of q', $q'(\mathbf{x}) = \sum_{j=1}^{3} c_{2j} q'(j\mathbf{x}/3)$ where the coefficients c_{2j} are defined by (4) (with $m = 3$). In virtue of (4), we also have that the polynomial $p'(\mathbf{x}) = \sum_{j=1}^{3} c_{2j}\, p(j\mathbf{x}/3)$ is 2-homogeneous. Hence

$$\|q'(\mathbf{x}) - p'(\mathbf{x})\| \leqslant \sum_{j=1}^{3} |c_{2j}| \left(w_3(X', Y) + \delta\right) \|\mathbf{x}\|^2 = C \left(w_3(X', Y) + \delta\right) \|\mathbf{x}\|^2$$

for all $\mathbf{x} \in X'$.

Since p' is locally bounded, we can express it in the form $p'(\mathbf{x}) = \Psi(\mathbf{x},\mathbf{x})$, where $\Psi: X' \times X' \longrightarrow Y$ is a continuous symmetric bilinear form. Consequently, there is a bounded linear operator $S: X' \longrightarrow L(X', Y)$ such that $(S\mathbf{x})\mathbf{y} = \Psi(\mathbf{x},\mathbf{y})$. It follows that $(S\mathbf{x})\mathbf{y} = (S\mathbf{y})\mathbf{x}$. Define bounded linear operators S_{11}, S_{12}, S_{21} and S_{22} from X into $L(X, Y)$ by

$$\begin{aligned}
(S_{11}x)y &= (S(x,0))(y,0),\\
(S_{12}x)y &= (S(x,0))(0,y),\\
(S_{21}x)y &= (S(0,x))(y,0),\\
(S_{22}x)y &= (S(0,x))(0,y)
\end{aligned}$$

for $x, y \in X$. Then $(S_{21}x)y = (S_{12}y)x$, and for every $\varepsilon_1, \varepsilon_2 = \pm 1$ we have

$$\begin{aligned}
&\left\| \tfrac{1}{6} g(x_1)x_2 - \varepsilon_1\varepsilon_2 \sum_{j,k \leqslant 2} \varepsilon_j \varepsilon_k (S_{jk}x_j)x_k \right\| \\
&\quad = \| q'((\varepsilon_1 x_1, \varepsilon_2 x_2)) - \Psi((\varepsilon_1 x_1, \varepsilon_2 x_2),\ (\varepsilon_1 x_1, \varepsilon_2 x_2)) \| \\
&\quad \leqslant C \left(w_3(X', Y) + \delta \right) \|\mathbf{x}\|^2_{X'} \\
&\quad \leqslant 2C \left(w_3(X', Y) + \delta \right) (\|x_1\|^2_X + \|x_2\|^2_X).
\end{aligned}$$

Averaging over choices of sign, we obtain

$$\|(\tfrac{1}{6} g(x_1) - 2(S_{12}x_1))x_2\| \leqslant 2C \left(w_3(X', Y) + \delta \right) (\|x_1\|^2 + \|x_2\|^2), \qquad x_1, x_2 \in X.$$

This leads to the desired inequality

$$\|g(x) - 12S_{12}x\| \leqslant 24C \left(w_3(X', Y) + \delta \right) \|x\|, \qquad x \in X$$

by putting $x_1 = x$ and $\|x_2\| = \|x\|$. □

Proof of Proposition 8. Let $f \in \mathcal{B}(B_X,\ L(X, Y))$ and $\omega_2(f) \leqslant 1$. By translation, we can assume that $f(0) = 0$. Set $g(x) = \|x\| \left(f\left(\frac{x}{2\|x\|}\right) - f\left(-\frac{x}{2\|x\|}\right) \right)$ for every $x \neq 0 \in X$, $g(0) = 0$. Clearly, g is 1-homogeneous. It is easy to check that

$$\|f(x) - g(x)\| \leqslant 2\|x\| + 2 \tag{6}$$

and

$$\|g(x+y) - g(x) - g(y)\| \leqslant 11(\|x\| + \|y\|) \leqslant 22 \tag{7}$$

for $x, y \in B_X$ (see, for example, [V, Lemma 3.7 and proof of Proposition 3.6]). It follows from (7) and Lemma 9 that there are an absolute constant K and a bounded linear operator $F_\delta: X \longrightarrow L(X, Y)$ so that

$$\|g(x) - F_\delta x\| \leqslant K \left(w_3(X', Y) + \delta \right) \|x\|$$

for any $\delta > 0$ and for all $x \in X$. Restricting to $x \in B$ and using (6) give the desired inequality

$$\|f(x) - F_\delta x\| \leqslant C \left(w_3(X', Y) + \delta \right), \qquad x \in B. \qquad \square$$

Remark 10. (i) Again, an inspection of the above proof gives

$$w_2^l(X,\ L(X,Y)) \leqslant C w_3^l(X',Y)$$

with $X' = (X \oplus X)_p$ for some $p \in [1,\infty]$.

(ii) In particular, it follows from Proposition 8 and estimates obtained by Brudnyi and Kalton [BK] that for any n-dimensional X

$$w_2(X,X^*) \leqslant C w_3(X',\mathbf{R}) \leqslant C_1 \min\{\sqrt{n}, T_2(X)^2\} \log(n+1).$$

Brudnyi and Kalton also obtained that
(1) $w_2(l_p^n,\mathbf{R}) \leqslant 1602$ for $2 \leqslant p \leqslant \infty$ (see [BK, Theorem 3.9(c)]);
(2) $w_3(l_p^n,\mathbf{R}) \sim \log(n+1)$ for $2 \leqslant p < \infty$, and $c\ \log(n+1) \leqslant w_3(l_\infty^n,\mathbf{R}) \leqslant C(\log(n+1))^2$ (see [BK, Theorem 4.3]);
(3) $w_m(l_1^n,\mathbf{R}) \sim \log(n+1)$ for any $m \geqslant 2$ (see [BK, Corollary 5.7]).

Thus by Proposition 8 and Theorem 5, we have

$$w_2\left(l_p^n,\ \left(l_p^n\right)^*\right) \sim w_3(l_p^n,\mathbf{R}) \sim \log(n+1) \qquad \text{if } p = 1 \text{ or } 2 \leqslant p < \infty$$

(observe that $w_2(l_1^n, l_\infty^n) = w_2(l_1^n,\mathbf{R})$) and

$$c\ \log(n+1) \leqslant w_2(l_\infty^n, l_1^n) \leqslant C(\log(n+1))^2.$$

(It was obtained in [V, Propositions 3.6 and 4.25] by another way that $w_2(l_2^n, l_2^n) \sim \log(n+1)$.)

Problem 11. Is it true that $w_2(X,X^*) \sim w_3(X,\mathbf{R})$ for all X?

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